Two-way Source Coding Through a Relay

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***Abstract*— A 3-node lossy source coding problem for a 2-DMS (X_1, X_2) is considered. Source nodes 1 and 2 observe X_1 and X_2, respectively, and each wishes to reconstruct the other source with a prescribed distortion. To achieve these goals, nodes 1 and 2 send descriptions of their sources to relay node 3. The relay node then broadcasts a joint description to the source nodes. A cutset outer bound and a compress–linear code inner bound are established and shown to coincide in several special cases. A compute–compress inner bound is then presented and shown to outperform the compress–linear code in some cases. An outer bound based on Kaspi's converse for the two-way source coding problem is shown to be strictly tighter than the cutset outer bound.**

I. Introduction

Consider the two-way source coding through a relay problem depicted in Figure 1. Source node $j = 1, 2$ observes a discrete memoryless source (DMS) X_j and sends a description of its source to relay node 3. The relay node then broadcasts a message based on what it has received from nodes 1 and 2 so that node 1 can recover X_2 with distortion D_2 and node 2 can recover X_1 with distortion D_1. When the sources are independent and maximally compressed, i.e., node $j = 1, 2$ observes message M_j uniformly distributed over $[1 : 2^{nR_j}]$ and wishes to recover the other message losslessly, the optimal coding scheme involves linear network coding [1], [2]. Node $j = 1, 2$ transmits its message M_j at rate R_j. The relay then expresses each message as a binary sequence and broadcasts the modulo-2 sum of the two sequences to nodes 1 and 2. Upon receiving the modulo-2 sum, node j recovers the message of the other node by performing modulo-2 sum on the binary expression of its message and the received sequence. The required broadcast rate is $R_3 \geq \max\{R_1, R_2\}$, which coincides with the cutset lower bound.

In this paper we investigate the lossy two-way source coding through a relay problem. Unlike the lossless case, the rate distortion region is not known in general. We establish a cutset outer bound and a compress–linear code inner bound on the rate distortion region that coincide in some special cases. For example, when the sources are independent, cascading point-to-point lossy source coding and the above linear network coding scheme is optimal. If the sources are Gaussian, the optimal scheme is to replace point-to-point source coding with Wyner–Ziv coding [3]. We show that neither bound is tight in general. We then show that the relay broadcast rate can be strictly improved via a compute–compress scheme whereby the relay first computes a function of the sources losslessly and then broadcasts a description of the function to the source nodes.

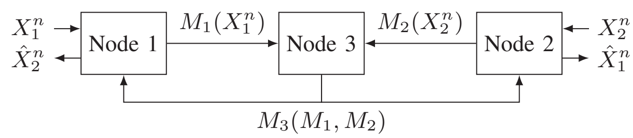

Fig. 1. Two-way source coding through a relay.

Several variations on the two-way source coding through a relay setting have been investigated. For example, in distributed lossy source coding, the broadcast rate $R_3 = 0$ and the goal is to recover both sources only at node 3. The lossless case is solved by the Slepian–Wolf coding [4], but the lossy case [5] remains open. Another variation is the cascade source coding [6], [7] where node 3 observers a DMS X_3, the rate $R_2 = 0$, and the goal is to recover the source X_1 or a function of X_1 and X_3 at node 2. The rate–distortion region for this case is not known in general. In these variations, there is no broadcast constraint on the relay node, which is motivated by wireless and satellite communications [2], [8]. The complementary delivery problem in [9] is similar to our setting except that node 3 has access to the sources X_1 and X_2, and thus $R_1 = R_2 = 0$. There is no longer a tradeoff between transmission rates and the rate–distortion function is known. If there is no relay and the two source nodes interactively communicate messages in multiple rounds, the rate–distortion region is known [10]. A channel coding setting for independent and maximally compressed sources are considered in [11]. Nodes 1 and 2 transmit messages to the relay through a discrete memoryless multiple access channel, and the relay sends a message to nodes 1 and 2 through a discrete memoryless broadcast channel. It is shown that joint network coding and relaying achieves higher capacity than traditional routing under some channel conditions.

In the next section, we formally define the problem. In Section III, an cutset outer bound is established. The compress–linear code inner bound is presented in Section IV and is shown to be tight in some special cases. The compute–compress inner bound is presented in Section V. In Section VI, we tighten the cutset outer bound via Kaspi's converse technique [10] for the two-way source coding problem. The notation and basic definitions follow [12].

II. Problem Formulation

A $(2^{nR_1}, 2^{nR_2}, 2^{nR_3}, n)$ *code* for the two-way source coding through a relay problem with 2-DMS (X_1, X_2) and distortion measures d_1 and d_2 consists of:

1) Two source encoders: Encoder $j = 1, 2$ assigns an index $m_j(x_j^n) \in [1 : 2^{nR_j})$ to each source sequence $x_j^n \in \mathcal{X}_j^n$.
2) A relay encoder that assigns an index $m_3(m_1, m_2) \in [1 : 2^{nR_3})$ to each index pair $(m_1, m_2) \in [1 : 2^{nR_1}) \times [1 : 2^{nR_2})$.
3) Two decoders: Decoder $j = 1, 2$ assigns an estimate $\hat{x}_{3-j}^n(m_3, x_j^n) \in \hat{\mathcal{X}}_{3-j}^n$ to each pair $(m_3, x_j^n) \in [1 : 2^{nR_3}) \times \mathcal{X}_j^n$.

A rate triple (R_1, R_2, R_3) is said to be *achievable* with distortion pair (D_1, D_2) if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, 2^{nR_3}, n)$ codes such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\left(d_j(X_j, \hat{X}_j)\right) \le D_j \text{ for } j = 1, 2.$$

The *rate distortion region* $\mathscr{R}(D_1, D_2)$ is the closure of rate triples (R_1, R_2, R_3) that are achievable with distortion pair (D_1, D_2).

We need the following definitions for later use. The rate distortion function for source X_1 is

$$R_1(D) = \min_{p(\hat{x}_1|x_1):\mathrm{E}(d_1(X_1,\hat{X}_1))\le D} I(X_1; \hat{X}_1).$$

When the side information X_2 is available at both the encoder and the decoder, the conditional rate distortion function for source X_1 is

$$R_{1|2}(D) = \min_{p(\hat{x}_1|x_1,x_2):\mathrm{E}(d_1(X_1,\hat{X}_1))\le D} I(X_1; \hat{X}_1|X_2).$$

When the side information X_2 is available only at the decoder, the Wyner–Ziv rate distortion function [3] for source X_1 is

$$R_{1|2}^{\mathrm{WZ}}(D) = \min_{p(u|x_1),\hat{x}_1(u,x_2):\mathrm{E}(d_1(X_1,\hat{X}_1))\le D} I(X_1; U|X_2).$$

The above three rate distortion functions satisfy $R_1(D) \ge R_{1|2}^{\mathrm{WZ}}(D) \ge R_{1|2}(D)$. Similarly, we can define $R_2(D)$, $R_{2|1}(D)$, and $R_{2|1}^{\mathrm{WZ}}(D)$ for source X_2 and side information X_1.

III. Cutset Outer Bound

Consider the cut between node 1 and the "super-node" consisting of nodes 2 and 3. By the Wyner–Ziv theorem [3], the transmission rate R_1 is lower bounded by $R_{1|2}^{\mathrm{WZ}}(D_1)$. To lower bound the broadcast rate R_3 of the relay, we consider a larger set of relay encoders $\tilde{m}_3(x_1^n, x_2^n)$. Note that every relay encoder $m_3(m_1(x_1^n), m_2(x_2^n)) = \tilde{m}(x_1^n, x_2^n)$ for some $\tilde{m}$. We again consider the cut between node 1 and the super-node. Now the side information X_1^n is available at both the encoder and the decoder. Thus, $R_3 \ge R_{2|1}(D_2)$. Similarly, we can establish cutset bounds for the cuts between node 2 and the remaining two nodes. Combining these bounds, we obtain the following cutset outer bound.

Theorem 1: Any achievable rate triple (R_1, R_2, R_3) for distortion pair (D_1, D_2) must satisfy the conditions

$$R_1 \ge R_{1|2}^{\mathrm{WZ}}(D_1),$$
$$R_2 \ge R_{2|1}^{\mathrm{WZ}}(D_2),$$
$$R_3 \ge \max\{R_{1|2}(D_1), R_{2|1}(D_2)\}.$$

IV. Compress–Linear Code Inner Bound

We establish the compress–linear code inner bound, where each source node sends a description of its source using Wyner–Ziv coding and the relay performs linear network coding. We show that this inner bound coincides with the cutset outer bound in several special cases.

A. Compress–linear code inner bound

We first consider a simple achievability scheme that uses routing at the relay. Nodes 1 and 2 use Wyner–Ziv coding at rates $R_{1|2}^{\mathrm{WZ}}(D_1)$ and $R_{2|1}^{\mathrm{WZ}}(D_2)$, respectively, to send descriptions of their own sources to the relay. The relay then broadcasts the received indices (M_1, M_2). Clearly by the Wyner–Ziv theorem the distortion constraints are satisfied. The routing inner bound is the set of rate triples (R_1, R_2, R_3) satisfying $R_1 \ge R_{1|2}^{\mathrm{WZ}}(D_1)$, $R_2 \ge R_{2|1}^{\mathrm{WZ}}(D_2)$ and $R_3 \ge R_{1|2}^{\mathrm{WZ}}(D_1) + R_{2|1}^{\mathrm{WZ}}(D_2)$. In the following theorem, we show that the relay broadcast rate in the routing inner bound can be reduced by exploiting the broadcast capability of the relay based on linear network coding.

Theorem 2: The *compress–linear code inner bound* on the rate distortion region $\mathscr{R}(D_1, D_2)$ consists of the set of rate triples (R_1, R_2, R_3) such that

$$R_1 \ge R_{1|2}^{\mathrm{WZ}}(D_1),$$
$$R_2 \ge R_{2|1}^{\mathrm{WZ}}(D_2),$$
$$R_3 \ge \max\{R_{1|2}^{\mathrm{WZ}}(D_1), R_{2|1}^{\mathrm{WZ}}(D_2)\}.$$

Proof: Node $j = 1, 2$ encodes source X_j^n into an index M_j using Wyner–Ziv coding at rate $r_j = R_{j|3-j}^{\mathrm{WZ}}(D_j)$. Let $U_j^{r_j}$ be the r_j-bit binary expression of index M_j. Without loss of generality, assume that $r_1 \ge r_2$. Upon receiving both indices M_1 and M_2, the relay appends $(r_1 - r_2)$ zeros to $U_2^{r_2}$ to obtain an r_1-bit binary sequence $\tilde{U}_2^{r_1}$. Then the relay broadcasts the index M_3 corresponding to $U_3^{r_1} = U_1^{r_1} \oplus \tilde{U}_2^{r_1}$ at rate r_1. Nodes 1 and 2 first recover M_2 and M_1 respectively by computing $U_1^{r_1} \oplus U_3^{r_1}$ and $\tilde{U}_2^{r_1} \oplus U_3^{r_1}$ and then use Wyner–Ziv decoding. Therefore, the required relay broadcast rate is $\max\{r_1, r_2\} = \max\{R_{1|2}^{\mathrm{WZ}}(D_1), R_{2|1}^{\mathrm{WZ}}(D_2)\}$. ■

Note that the broadcast rate of the routing inner bound can be reduced by a factor of 2 in the worst case. In the following subsection, we show that the compress–linear code inner bound is tight for some special cases. In Section V, we show that the compress–linear code inner bound is not tight in general.

B. Special Cases

We consider four special cases where the compress–linear code inner bound coincides with the cutset outer bound.

1) Recovering both sources losslessly: Suppose that nodes 1 and 2 wish to recover the source of the other node losslessly, that is, distortion measures d_1 and d_2 are Hamming distortions and $D_1 = D_2 = 0$. Then the Wyner–Ziv rate distortion function and the conditional rate distortion function are equal, and the compress–code inner bound and the cutset outer bound become

$$\begin{aligned} R_1 &\geq H(X_1|X_2), \\ R_2 &\geq H(X_2|X_1), \\ R_3 &\geq \max\{H(X_1|X_2), H(X_2|X_1)\}. \end{aligned}$$

The rate region for a generalization of this lossless example to a network with three source nodes is investigated in [8].

2) Independent sources: If the sources X_1 and X_2 are independent, then the side-information cannot reduce compression rates. Thus, nodes 1 and 2 can simply perform point-to-point lossy source coding independently, and the relay uses linear network coding. This yields the rate distortion region

$$\begin{aligned} R_1 &\geq R_1(D_1), \\ R_2 &\geq R_2(D_2), \\ R_3 &\geq \max\{R_1(D_1), R_2(D_2)\}. \end{aligned}$$

Note that the two-way source coding through a relay example in [2] is a special case of the above two cases, where the sources are independent and maximally compressed and nodes 1 and 2 wish to exchange their sources losslessly.

3) 2-WGN sources: When the sources are two correlated white Gaussian noise processes (2-WGN) with average powers P_1 and P_2, respectively, and correlation coefficient ρ, the rate distortion functions for side-information only at the decoder and for side-information at both the encoder and the decoder are the same. Thus, the compress–linear code inner bound and the cutset outer bound coincide and are equal to

$$\begin{aligned} R_1 &\geq \mathrm{R}\left(P_1(1-\rho^2)/D_1\right), \\ R_2 &\geq \mathrm{R}\left(P_2(1-\rho^2)/D_2\right), \\ R_3 &\geq \max\left\{\mathrm{R}\left(P_1(1-\rho^2)/D_1\right), \mathrm{R}\left(P_2(1-\rho^2)/D_2\right)\right\}, \end{aligned}$$

where $\mathrm{R}(x) = \max\{(1/2)\log x, 0\}$.

4) Recovering X_2 losslessly: In this case, the distortion measure d_2 is Hamming distortion and $D_2 = 0$. The compress–linear code inner bound reduces to

$$\begin{aligned} R_1 &\geq R^{\mathrm{WZ}}_{1|2}(D_1), \\ R_2 &\geq H(X_2|X_1), \\ R_3 &\geq \max\{R^{\mathrm{WZ}}_{1|2}(D_1), H(X_2|X_1)\}, \end{aligned}$$

and the cutset outer bound reduces to

$$\begin{aligned} R_1 &\geq R^{\mathrm{WZ}}_{1|2}(D_1), \\ R_2 &\geq H(X_2|X_1), \\ R_3 &\geq \max\{R_{1|2}(D_1), H(X_2|X_1)\}. \end{aligned}$$

These two bounds are not tight in general. However, they coincide if $H(X_2|X_1) \geq R^{\mathrm{WZ}}_{1|2}(D_1)$ or X_2 is a function of X_1. The former follows by $R^{\mathrm{WZ}}_{1|2}(D_1) \geq R_{1|2}(D_1)$, and the latter follows by the fact that X_2 becomes side-information at both the encoder node 1 and the decoder node 2 and thus the Wyner–Ziv rate distortion function $R^{\mathrm{WZ}}_{1|2}(D_1)$ is replaced with the conditional rate distortion function $R_{1|2}(D_1)$.

V. Compute–compress Inner Bound

In previous subsection, we showed that the compress–linear code inner bound coincides with the cutset outer bound in some special cases. However, the relay broadcast rate of the compress–linear code bound is in general higher than the cutset outer bound on the broadcast rate. In the following example, we consider a compute–compress inner bound of which the relay broadcast rate is the same as the cutset bound.

Example 1: Let (X_1, X_2) be doubly symmetric binary sources (DSBS) with parameter $0 < p < 1/2$, that is, $X_1 = X_2 \oplus Z$ where $X_2 \sim \mathrm{Bern}(1/2)$ and $Z \sim \mathrm{Bern}(p)$ are independent. The distortion measures d_1 and d_2 are Hamming distortions. Node $j = 1, 2$ first sends a description of its source X_j to the relay such that the relay can compute the modulo-two sum $V(X_1, X_2) = X_1 \oplus X_2$ losslessly. The relay then uses point-to-point rate distortion codes to send a description of the modulo-2 sum to both nodes. Using random linear codes [13], (R_1, R_2) is achievable provided

$$R_1 \geq H(V(X_1, X_2)|X_2) = H(p), \tag{1}$$

$$R_2 \geq H(V(X_1, X_2)|X_1) = H(p), \tag{2}$$

where $H(p) = -p\log p - (1-p)\log(1-p)$, and the required broadcast rate is

$$\begin{aligned} R_3 &\geq \max\{R_{1|2}(D_1), R_{1|2}(D_2)\} \\ &= \max\{H(p) - H(D_1), H(p) - H(D_2)\}. \end{aligned}$$

Note that the constraint on the broadcast rate here is the same as the cutset bound. In this example, the compute–compress inner bound and the compress–linear code inner bound do not coincide with each other since the rate triple $\left(R^{\mathrm{WZ}}_{1|2}(D_1), R^{\mathrm{WZ}}_{1|2}(D_2), \max\{R^{\mathrm{WZ}}_{1|2}(D_1), R^{\mathrm{WZ}}_{1|2}(D_2)\}\right)$ lies only in the compress–linear code inner bound, and the rate triple $\left(H(p), H(p), \max\{R_{1|2}(D_1), R_{1|2}(D_2)\}\right)$ lies only in the compute–compress inner bound. By time sharing between these two inner bounds, we can obtain an inner bound that is larger than each bound.

In general, the optimal rate region for multiterminal lossless computing is not known, and the rate region similar to (1) and (2) is an outer bound. If the function to be computed satisfies certain conditions, the Slepian–Wolf rate region is optimal [14]. In the following theorem, we present a general compute–compress inner bound. For some special cases such as the DSBS example above, the required rates for the compute phase can be reduced.

Theorem 3: The *compute–compress* inner bound on the rate distortion region $\mathscr{R}(D_1, D_2)$ consists of the set of rate triple

(R_1, R_2, R_3) such that

$$\begin{aligned} R_1 &\geq I(X_1; U_1|X_2, Q), \\ R_2 &\geq I(X_2; U_2|X_1, Q), \\ R_1 + R_2 &\geq I(X_1, X_2; U_1, U_2|Q), \\ R_3 &\geq I(V; W|X_1, Q), \\ R_3 &\geq I(V; W|X_2, Q) \end{aligned}$$

for some $p(q)p(u_1|x_1, q)p(u_2|x_2, q)p(w|v, q)$ and functions $v(x_1, x_2)$, $\hat{x}_1(w, x_2)$, and $\hat{x}_2(w, x_1)$ such that $H(V(X_1, X_2)|U_1, U_2) = 0$ and

$$\begin{aligned} \mathrm{E}\left(d_1(X_1, \hat{X}_1(W, X_2))\right) &\leq D_1, \\ \mathrm{E}\left(d_2(X_2, \hat{X}_2(W, X_1))\right) &\leq D_2. \end{aligned}$$

Proof outline: Nodes 1 and 2 use Berger–Tung coding to encode the sources so that the relay can recover (U_1, U_2). Since $H(V(X_1, X_2)|U_1, U_2) = 0$, the relay can compute the function $V(X_1, X_2)$ based on (U_1, U_2). The relay then uses Wyner–Ziv coding by first covering V by W and then sending a description of W via binning. ■

VI. General Outer Bound

In Section III, we established the cutset outer bound by allowing the relay encoder to have access to X_1^n and X_2^n, i.e., its index is a function of (X_1^n, X_2^n) instead of (M_1, M_2). Sending X_1^n and X_2^n to the relay, however, may require much higher rates than the Wyner–Ziv coding rates in the cutset bound. Thus, as we will show the cutset outer bound can be strictly loose. In the following, we establish a tighter outer bound.

Theorem 4: Any rate triple (R_1, R_2, R_3) achievable with distortion pair (D_1, D_2) must satisfy

$$\begin{aligned} R_1 &\geq I(X_1; U_1|X_2), \\ R_2 &\geq I(X_2; U_2|X_1), \\ R_3 &\geq I(X_1; V|X_2, U_2), \\ R_3 &\geq I(X_2; V|X_1, U_1) \end{aligned}$$

for some $p(u_1, u_2|x_1, x_2)p(v|u_1, u_2)$, $\hat{x}_1(v, x_2, u_2)$, and $\hat{x}_2(v, x_1, u_1)$ such that $U_1 \to X_1 \to X_2$, $U_2 \to X_2 \to X_1$, $V \to (X_1, U_2) \to X_2$, $V \to (X_2, U_1) \to X_1$, and

$$\begin{aligned} \mathrm{E}\left(d_1(X_1, \hat{X}_1(V, X_2, U_2))\right) &\leq D_1, \\ \mathrm{E}\left(d_2(X_2, \hat{X}_2(V, X_1, U_1))\right) &\leq D_2. \end{aligned}$$

The proof is based on the converse in [10] and is given in the Appendix. Now we revisit Example 1. Suppose that distortion $D_2 = 1$, that is, node 1 does not need to recover source X_2. The cutset outer bound for distortion pair $(D_1, 1)$ is equal to

$$\begin{aligned} R_1 &\geq R_{1|2}^{\mathrm{WZ}}(D_1), \\ R_3 &\geq R_{1|2}(D_1), \end{aligned}$$

and the outer bound in Theorem 4 is equal to

$$\begin{aligned} R_1 &\geq I(X_1; U_1|X_2), \\ R_2 &\geq I(X_2; U_2|X_1), \\ R_3 &\geq I(X_1; V|X_2, U_2), \end{aligned}$$

for some $p(u_1, u_2, v|x_1, x_2) = p(u_1, u_2|x_1, x_2)p(v|u_1, u_2)$ and $\hat{x}_1(v, x_2, u_2)$ such that

$$\mathrm{E}\left(d_1(X_1, \hat{X}_1(V, X_2, U_2))\right) \leq D_1.$$

Now consider the rate distortion region when $R_2 = 0$. For the outer bound in Theorem 4, U_2 needs to satisfy $I(X_2; U_2|X_1) = 0$, i.e., $U_2 \to X_1 \to X_2$. But since $U_2 \to X_2 \to X_1$ also form a Markov chain, $p(u_2|x_1, x_2) = p(u_2|x_1) = p(u_2|x_2)$. Furthermore, $p(x_1, x_2) > 0$ for all (x_1, x_2), and thus for any $(x_1, x_2) \neq (\tilde{x}_1, \tilde{x}_2)$,

$$\begin{aligned} p(u_2|x_1, x_2) &= p(u_2|x_1) = p(u_2|x_1, \tilde{x}_2) \\ &= p(u_2|\tilde{x}_2) = p(u_2|\tilde{x}_1, \tilde{x}_2), \end{aligned}$$

that is, U_2 is independent of (X_1, X_2). This implies that the bound on R_3 can be expressed as

$$R_3 \geq I(X_1; V|X_2, U_2) = I(X_1; V, U_2|X_2) \geq R_{1|2}^{\mathrm{WZ}}(D_1).$$

Furthermore, by the data processing inequality, the bound on R_1 becomes

$$R_1 \geq I(X_1; U_1|X_2) \geq I(X_2; V, U_2|X_2) \geq R_{1|2}^{\mathrm{WZ}}(D_1).$$

Therefore, the rate triple $(R_{1|2}^{\mathrm{WZ}}(D_1), 0, R_{1|2}(D_1))$ in the cutset outer bound is not achievable, and the cutset bound is strictly loose in this DSBS example.

VII. Conclusion

We established inner and outer bounds on the rate distortion region for the two-way source coding through a relay problem that coincide in some special cases. In the compress–linear code achievability scheme, the source node communication rates coincide with the cutset bound, but the relay communication rate is higher. The compute–compress scheme achieves lower relay communication rate by increasing the source node rates beyond the cutset bound. The rate distortion region is not known in general and there are several interesting directions to improve the inner bounds. Are there nontrivial cases where the compute–compress code is optimal? How do we combine the two achievability schemes beyond time sharing? It would also be interesting to investigate inner bounds for the multi-round version of the problem. The outer bound can be readily extended to this case.

APPENDIX

Proof of Theorem 4: We first bound the transmission rate of R_1 by the following chain of inequalities.

$$\begin{aligned} nR_1 &\geq H(M_1) \geq H(M_1|X_2^n) = I(X_1^n; M_1|X_2^n) \\ &= \sum_{i=1}^{n} I(X_{1i}; M_1, X_1^{i-1}, X_2^{i-1}, X_{2,i+1}^n | X_{2i}) \\ &\geq \sum_{i=1}^{n} I(X_{1i}; U_{1i}|X_{2i}), \end{aligned}$$

where $U_{1i} = (M_1, X_1^{i-1}, X_{2,i+1}^n)$. Similarly, $nR_2 \geq \sum_{i=1}^n I(X_{2i}; U_{2i}|X_{1i})$, where $U_{2i} = (M_2, X_{1,i+1}^n, X_2^{i-1})$. Next we bound the relay broadcast rate.

$$\begin{aligned} nR_3 &\geq H(M_3) \geq H(M_3|M_1, X_1^n) = I(X_2^n; M_3|M_1, X_1^n) \\ &= \sum_{i=1}^{n} I(X_{2i}; M_3, X_{1,i+1}^n | M_1, X_1^{i-1}, X_{1i}, X_{2,i+1}^n) \\ &\geq \sum_{i=1}^{n} I(X_{2i}; V_i | X_{1i}, U_{1i}), \end{aligned}$$

where $V_i = M_3$. Similarly, $nR_3 \geq \sum_{i=1}^n I(X_{1i}; V_i|X_{2i}, U_{2i})$.

Now we claim that for every $i \in [1:n]$, (i)

$$\begin{aligned} &\mathrm{E}\left(d_1(X_{1i}, \hat{x}_1^*(i, V_i, X_{2i}, U_{2i}))\right) \leq \mathrm{E}\left(d_1(X_{1i}, \hat{x}_{1i}(M_3, X_2^n)\right), \\ &\mathrm{E}\left(d_2(X_{2i}, \hat{x}_2^*(i, V_i, X_{1i}, U_{1i}))\right) \leq \mathrm{E}\left(d_2(X_{2i}, \hat{x}_{2i}(M_3, X_1^n)\right) \end{aligned}$$

for some functions $\hat{x}_1^*$ and $\hat{x}_2^*$, and (ii) $V_i \to (U_{1i}, U_{2i}) \to (X_{1i}, X_{2i})$, $V_i \to (U_{1i}, X_{2i}) \to X_{1i}$, and $V_i \to (U_{2i}, X_{1i}) \to X_{2i}$. To prove these two claims, we use the technique in [15] to verify Markovity. Consider the factorization of distribution

$$\begin{aligned} p(x_1^n, x_2^n, m_1, m_2, m_3) &= p(x_1^{i-1}, x_2^{i-1}) p(x_{1i}, x_{2i}) \\ &\quad \cdot p(x_{1,i+1}^n, x_{2,i+1}^n) p(m_1|x_1^n) p(m_2|x_2^n) p(m_3|m_1, m_2). \end{aligned}$$

and the corresponding undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in Figure 2, where $\mathcal{V}$ is the set of vertices and an edge $(V_1, V_2) \in \mathcal{E}$ if v_1 and v_2 are in some common factor. We can verify the Markov chain $V_i \to (U_{1i}, U_{2i}) \to (X_{1i}, X_{2i})$ by showing that every path in the graph from V_i to (X_{1i}, X_{2i}) must pass through (U_{1i}, U_{2i}). Similarly, we can check the other Markov chains in the second claim and also show that $X_{2,i+1}^n \to (X_{2i}, U_{2i}, V_i) \to X_{1i}$, which we will need to prove the first claim in the following. Consider the expected distortion

$$\begin{aligned} &\mathrm{E}\left(d_1(X_{1i}, \hat{x}_{1i}(M_3, X_2^n)\right) \\ &\quad = \sum p(x_{1i}^n, x_2^n, m_2, m_3) d_1(x_{1i}, \hat{x}_{1i}(m_3, x_2^n)) \\ &\quad = \sum p(x_{2i}^n, u_{2i}, v_i) p(x_{1i}|x_{2i}^n, u_{2i}, v_i) \\ &\qquad \cdot d_1(x_{1i}, \hat{x}'_{1i}(x_{2i}^n, u_{2i}, v_i)) \\ &\quad = \sum p(x_{2i}^n, u_{2i}, v_i) p(x_{1i}|x_{2i}, u_{2i}, v_i) \\ &\qquad \cdot d_1(x_{1i}, \hat{x}'_{1i}(x_{2i}^n, u_{2i}, v_i)), \end{aligned}$$

where the second equality follows by defining $\hat{x}'_{1i}(x_{2i}^n, u_{2i}, v_i) = \hat{x}_{1i}(m_3, x_2^n)$ for all $x_{1,i+1}^n$, and the last step

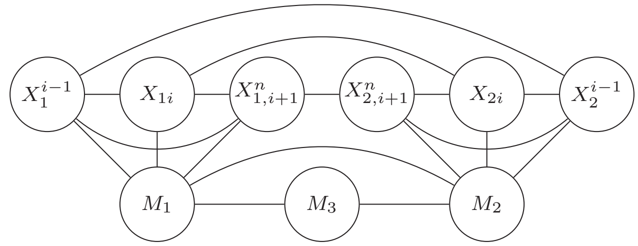

Fig. 2. The graph of the distribution $p(x_1^n, x_2^n, m_1, m_2, m_3)$.

follows by the Markov chain $X_{2,i+1}^n \to (X_{2i}, U_{2i}, V_i) \to X_{1i}$. Let $\hat{x}_1^*(i, v, x_2, u_2) = \hat{x}'_{1i}(x_{2i}, x_{2,i+1}^{n*}, u_{2i}, v_i)$, where

$$\begin{aligned} x_{2,i+1}^{n*}(x_{2i}^n, u_{2i}, v_i) = \arg\min_{x_{2,i+1}^n} \sum_{x_{1i}} p(x_{1i}|x_{2i}, u_{2i}, v_i) \\ \cdot d_1(x_{1i}, \hat{x}'_{1i}(x_{2i}^n, u_{2i}, v_i)). \end{aligned}$$

Then

$$\mathrm{E}\left(d_1(X_{1i}, \hat{x}_1^*(i, V_i, X_{2i}, U_{2i}))\right) \leq \mathrm{E}\left(d_1(X_{1i}, \hat{x}_{1i}(M_3, X_2^n)\right).$$

Similarly, there exists some function $\hat{x}_2^*(i, v, x_1, u_1)$ such that

$$\mathrm{E}\left(d_2(X_{2i}, \hat{x}_2^*(i, V_i, X_{1i}, U_{1i}))\right) \leq \mathrm{E}\left(d_2(X_{2i}, \hat{x}_{2i}(M_3, X_1^n)\right).$$

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